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The D1/D5 system and singular CFT

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Abstract: We study the conformal field theory of the D1/D5 system compactified on $\mathbf{X}$ ($\mathbf{X}$ is $\mathbf{T}^4$ or $\mathbf{K3}$). It is described by a sigma model whose target space is the moduli space of instantons on $\mathbf{X}$. For values of the parameters where the branes can separate, the spectrum of dimensions in the conformal field theory exhibits a continuum above a gap. This continuum leads to a pathology of the conformal field theory, which explains a variety of problems in various systems. In particular, we explain the apparent discrepancy between different methods of finding the spectrum of chiral fields at certain points in the moduli space of the system.

Keywords: Field Theories in Lower Dimensions, Superstring Vacua, Models of Quantum Gravity, Supersymmetry and Duality.

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Contents

1. Introduction

The D1/D5 system, which has been much studied of late, is constructed from branes in $\mathbb{R}^6 \times \mathbf{X}$, where $\mathbf{X}$ is $\mathbf{T}^4$ or $\mathbf{K3}$. One considers Q_5 D5-branes wrapped on X, making strings in $\mathbb{R}^6$; and one adds Q_1 D1-branes that are localized on $\mathbf{X}$ and parallel to the strings made from the fivebranes. If Q_1 and Q_5 are large, this system has a supergravity description as a "black string," whose near horizon geometry [1] looks like $AdS_3 \times \mathbf{S}^3 \times \mathbf{X}$. In this limit, the D1/D5 system is believed [2] to be described by a conformal field theory on the boundary of $AdS_3 \times \mathbf{S}^3$. (The boundary in question is a conformal boundary in the sense of Penrose [3].)

One can generalize the D1/D5 system by turning on various other fields or charges, such as theta angles, RR fields, or D3-branes wrapped on two-cycles in $\mathbf{X}$. However, the simplest D1/D5 system with no such extra fields has the basic property that the branes can separate at no cost in energy. In fact, any collection of D5 branes (wrapped on $\mathbf{X}$) and D1 branes (localized on $\mathbf{X}$), with all branes parallel in $\mathbb{R}^6$, is BPS-saturated (if the theta angles and RR fields are zero). Hence, the D1/D5 system can break into pieces at no cost in energy. The goal of the present paper is to study the implications of this for the boundary conformal field theory.

One description of the low energy physics of this system is provided by the $\mathrm{SU}(Q_5)$ gauge theory on the D5-branes. The D1-branes can be interpreted as Q_1 instantons in this gauge theory, and the D3-branes correspond to magnetic fluxes, representing a non-zero first Chern class. We denote by $\mathcal{M}$ the moduli space of $\mathrm{SU}(Q_5)$ instantons on $\mathbf{X}$ with onebrane charge[1] Q_1 and with c_1 determined by the number of threebranes. The dynamics of our system is described by a $(4,4)$ sigma model whose target space is $\mathcal{M}$.

To reproduce the "pure" D1/D5 system, we set $c_1 = 0$. Then we expect to see in the gauge theory that the branes can separate. In fact, an instanton can become "small," and separate from the D1/D5 system as a D1-brane. Or the structure group of a $\mathrm{SU}(Q_5)$ instanton might reduce to $\mathrm{SU}(Q_5') \times \mathrm{SU}(Q_5'')$ with instanton numbers Q_1' and Q_1'' in the two factors (and $Q_5' + Q_5'' = Q_5$, $Q_1' + Q_1'' = Q_1$). Then the two groups of branes, with respective quantum numbers (Q_1', Q_5') and (Q_1'', Q_5''), can separate in $\mathbb{R}^6$. A special case of this, with $(Q_1', Q_5') = (0, 1)$, is the emission of a fivebrane, a process that is T-dual to shrinking of an instanton and its emission as a D1-brane.

In any of these cases, the separation of the branes is described in terms of gauge theory by a passage from a Higgs to a Coulomb branch. To see this, one uses an effective description as a two-dimensional gauge theory on the string. For example, the emission of a small instanton is described by an effective $\mathrm{SU}(1)$ gauge theory in $1+1$ dimensions with Q_5 hypermultiplets of charge 1. This theory has a Higgs branch, describing a non-small instanton, and a Coulomb branch, describing a D1-brane that has been ejected from the D1/D5 system. The two branches meet at the small instanton singularity. Similarly, the splitting of the (Q_1, Q_5) system to (Q_1', Q_5') plus (Q_1'', Q_5'') means that the structure group of the instanton reduces from $\mathrm{SU}(Q_5)$ to $\mathrm{SU}(Q_5') \times \mathrm{SU}(Q_5'')$, leaving unbroken an extra $\mathrm{SU}(1)$. This $\mathrm{SU}(1)$ is coupled to several charged hypermultiplets (which represent the instanton moduli that must be set to zero to reduce the structure group of the instanton to $\mathrm{SU}(Q_5') \times \mathrm{SU}(Q_5'')$). The low energy theory describing the splitting is $\mathrm{SU}(1)$ coupled to these hypermultiplets.

In supersymmetric gauge theories above two dimensions, the Higgs and Coulomb branches parametrize families of supersymmetric vacua. In two dimensions, this is not so; the usual infrared divergences of massless bosons cause the quantum wave functions to spread out over the Higgs or Coulomb branches. Nevertheless, even in two dimensions, the Higgs and Coulomb branches are described by different conformal field theories [4]. The different branches have different R-symmetries and usually have different central charges.

So even though the two branches meet classically, they are disconnected in conformal field theory. How can this happen? One idea is that as one flows to the

[1] For $\mathbf{X} = \mathbf{T}^4$, the onebrane charge Q_1 equals the instanton number Q_1', while for $\mathbf{X} = \mathbf{K3}$, $Q_1 = Q_1' - Q_5$. We will loosely refer to Q_1 as the instanton number.

infrared, the metric on the two branches might be renormalized [5] so that the classical meeting place of the two branches would be "infinitely far away" as seen on either branch. Something like this happens to the Coulomb branch at the one-loop level for any values of Q_1 and Q_5 for which both a Coulomb and Higgs branch exist [6]. For a single vector multiplet, the classical Coulomb branch is a copy of $\mathbb{R}^4$; we regard the point of intersection with the classical Higgs branch as the origin of $\mathbb{R}^4$ and let u denote a radial variable on $\mathbb{R}^4$ that vanishes at the origin. The classical metric $du^2+u^2d\Omega^2$ of the Coulomb branch is renormalized at the one-loop level so that near $u=0$ it looks like

$$\frac{1}{u^2}\left(du^2+u^2d\Omega^2\right). \tag{1.1}$$

Thus $u=0$ is at infinite distance. Moreover, there is a "Liouville coupling" $R\ln u$ (with R the worldsheet scalar curvature) such that the string coupling constant diverges as one approaches $u=0$. The metric (1.1) describes an infinitely long tube near $u=0$, which is why we speak of "tubelike" behavior, and the string coupling diverges as one goes down the tube.

One might hope for something similar on the Higgs branch. But the fate of the Higgs branch must be more subtle. One cannot simply find a quantum correction generating a tubelike metric, since gauge loops do not renormalize the hyper-Kähler metric on the Higgs branch [7]. However, the description of the Higgs branch via a sigma model with target space the classical Higgs branch $\mathcal{M}$ is not good near the singularities of $\mathcal{M}$. One might hope that in terms of some new variables that do give an effective description near the singularity one would find a tubelike behavior.

This is known in certain special cases via duality between Type IIA at an $\mathbf{A}_{n-1}$ singularity and Type IIB with n parallel D5-branes [8, 9]. In more generality, such a description can be obtained using duality between the Higgs and Coulomb branches of $(4,4)$ supersymmetric gauge theories in two dimensions [10, 11].

The purpose of the present paper is to exhibit tubelike behavior near the singularities of the D1/D5 system, and therefore various other systems to which it is dual. We do this by elaborating upon a construction introduced by Maldacena, Michelson, and Strominger [12]. We will give a description of the physics of the Higgs branch near its singularity in terms of an effective two-dimensional field ϕ. ϕ will be a Liouville field with kinetic energy $|d\phi|^2$ and a Liouville coupling ϕR. The classical singularity of the Higgs branch, or in other words its meeting with the classical Coulomb branch, corresponds to $\phi=\infty$. Because of the Liouville term, the string coupling blows up as one goes to $\phi=\infty$.

Thus, whether one begins on the Coulomb branch or the Higgs branch, the meeting place of these two branches is at infinite distance in terms of the right variables. Hence, starting from either branch, one can never reach the other. Starting from either branch and trying to approach the second, one must go "down the tube" and the string coupling constant blows up. The blowup of the string coupling constant is extremely important in some applications of the systems that have this behavior (like Type IIA at an $\mathbf{A}_{n-1}$ singularity) because it makes it possible to have nonperturbative effects (in that case, enhanced gauge symmetry) that cannot be turned off by going to weak coupling.

Since the usual D1/D5 system is the quantum mechanics of a Higgs branch, the tubelike nature of the singularity of the Higgs branch has specific consequences for the boundary conformal field theory that governs this system. Liouville theory has a continuous spectrum of dimensions above a certain threshold. So the boundary conformal field theory will have this property. Also, in Liouville theory with a strong coupling end that is not cut off or protected in any way, correlation functions generally diverge from integration over the Liouville field. So correlation functions of the D1/D5 conformal field theory can be expected to receive divergent contributions near the singularity.

Using S duality, we can transform the problem to Q_5 NS5-branes and Q_1 fundamental strings [13]. In this case, one can study the system by using conformal field theory of the fundamental strings, and one can also hope to compare this to the boundary conformal field theory at infinity. However, such a comparison will be affected by the continuous spectrum of dimensions mentioned in the last paragraph. For example, above the threshold, chiral primary states lie in the continuum, and one should expect difficulty in counting them.

In this discussion, we have emphasized the AdS_3 examples. But part of our discussion is also relevant to AdS_n for $n > 3$.

The discussion in the present paper is reminiscent of "the membrane at the end of the universe" [14] and the Liouville theory of AdS_3 [15], though the interpretation seems to be somewhat different.

This paper is organized as follows. In section 2, we review the moduli space of the D1/D5 system, explaining more precisely what conditions on the moduli are needed to get the singularity that we will be studying.

In section 3, we show how the study of large strings or branes introduced in [12] gives an effective description of the singularity of the Higgs branch. We also show how, for AdS_n models with $n > 3$, one reproduces in this way expected properties of the boundary conformal field theories. Among other things, we show that the boundary must have positive scalar curvature for stability, in agreement with the known behavior of $N = 4$ super Yang-Mills theory in four dimensions.

In section 4, we analize more quantitatively the effective theory of the long string in AdS_3, and we determine the exact value of the threshold above which the conformal field theory has a continuous spectrum.

In section 5, we discuss the fate of the chiral states, and some additional applications.

JHEP04(1999)017

2. Near horizon moduli space and singularities

In this section, we will review the moduli space of AdS_3 compactifications, following [16], and then describe precisely where a singularity is expected.

2.1 Classification of models

Consider string compactification on $\mathbf{X} = \mathbf{T}^4$ or $\mathbf{K3}$. The duality group is $\mathcal{K} = \mathrm{SO}(5, n; \mathbb{Z})$, where $n = 5$ for $\mathbf{T}^4$ and $n = 21$ for $\mathbf{K3}$. The moduli space of vacua is

$$\mathcal{W} = \mathrm{SO}(5, n; \mathbb{Z})\backslash\mathrm{SO}(5, n; \mathbb{R})/\mathrm{SO}(5) \times \mathrm{SO}(n)\,. \tag{2.1}$$

We now want to consider a system consisting of parallel strings in $\mathbb{R}^6 \times \mathbf{X}$. The D1/D5 system is the special case in which the strings are either D1-branes or else D5-branes wrapped on $\mathbf{X}$. In general, the charge of a string is measured by a charge vector v that takes values in an even, unimodular lattice $\Gamma^{5,n}$ on which $\mathcal{K}$ acts. The quadratic form on $\Gamma^{5,n}$ has five positive and n negative eigenvalues. $\Gamma^{5,n}$ can be embedded in a vector space V which has metric of signature $(5, n)$. A point in $\mathcal{W}$ gives a decomposition $V = V_+ \oplus V_-$, where the quadratic form is positive on V_+ and negative on V_-. $\Gamma^{5,n}$ is analogous to a Narain lattice in string theory, and $V_\pm$ are analogous to the spaces of right-moving and left-moving momenta.

For a string with charge vector v to be a BPS configuration, it is necessary to have $v^2 \geq 0$. Since the lattice is even, we have

$$v^2 = 2N\,, \qquad \text{with } N \geq 0\,. \tag{2.2}$$

Given any vector v and any point in moduli space, we write $v = v_+ + v_-$, with $v_\pm \in V_\pm$. The tension of a string with charge v is then (up to a multiplicative constant independent of v and the moduli)

$$T(v) = |v_+|\,. \tag{2.3}$$

We are mainly interested in the case that the charge vector v is "primitive," in other words is not of the form $v = kv'$ with v' a lattice vector and k an integer greater than 1. The reason for this restriction is that otherwise the model is singular – capable of breaking into subsystems at no cost in energy – for all values of the moduli.

Now, any two primitive lattice vectors v and w with $v^2 = w^2 = 2N$ are equivalent up to a transformation by an element of $\mathcal{K}$, as explained in [16]. So up to a duality transformation, there is only one model for every positive integer N. $N = 0$ is the case of a single elementary string, so we are really interested in $N > 0$.

For example, the D1/D5 system is the case that there is a decomposition

$$\Gamma^{5,n} = \Gamma^{1,1} \oplus \Gamma^{4,n-1}\,, \tag{2.4}$$

where $\Gamma^{1,1}$ is a two-dimensional sublattice whose quadratic form in a suitable basis looks like

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.5}$$

and the moduli are such that the decomposition (2.4) commutes with the projection to V_+ and V_-. If the theta angles and RR fields vanish, then (as one can see in more detail from formulas given in [16]) there is such a decomposition with the D1 string and the string built from a wrapped D5 brane represented by null vectors $(1,0)$ and $(0,1)$ in $\Gamma^{1,1}$. The D1/D5 system can then be described by charges $(Q_1, Q_5) \in \Gamma^{1,1}$, and hence

$$N = Q_1 Q_5\,. \tag{2.6}$$

In this construction, v being primitive means that Q_1 and Q_5 are relatively prime.

As we have discussed in the introduction, the D1/D5 system is expected to have, for each value of Q_1 and Q_5, a singularity associated with a small $\mathrm{SU}(Q_5)$ instanton. To achieve such a singularity, one must adjust $4(Q_5-1)$ parameters in the instanton solution. The number of parameters that must be adjusted depends on Q_5, so the small instanton singularity of the D1/D5 system depends on the value of Q_5, and not on the product $N = Q_1Q_5$. Thus, a single system, characterized by the choice of one integer N, has different singularities corresponding to the different factorizations $N = Q_1Q_5$ with Q_1 and Q_5 relatively prime. To clarify this further, we will in section 2.2 analyze precisely where there are singularities of the near horizon theory.

Near horizon moduli space. Given a choice of charge vector v, one can construct a supergravity solution that describes a string of that charge. Here one finds [17] an interesting phenomenon: in the field of this string, the moduli are not constant, and vary in such a way that at the horizon, the vector v is "purely right-moving," that is, it lies in V_+. The moduli that would rotate V_+ and V_- so that v no longer lies in V_+ are "fixed scalars"; in the near horizon geometry of the string, they are massive.

The near horizon geometry therefore has a reduced moduli space. Roughly it is a Narain moduli space of signature $(4,n)$, since v is now constrained to lie in V_+ and only the four-dimensional orthocomplement of v in V_+ is free to vary. The near horizon geometry also has a reduced duality group, namely the subgroup $\mathcal{H}$ of $\mathcal{K}$ consisting of transformations that leave fixed the vector v. The moduli space of the near horizon geometry is

$$\mathcal{N} = \mathcal{H}\backslash \mathrm{SO}(4,n;\mathbb{R})/\mathrm{SO}(4)\times\mathrm{SO}(n)\,. \tag{2.7}$$

2.2 Location of singularities

Now we come to the question of under what condition the conformal field theory that describes the near horizon physics becomes singular. As we explained in the introduction, this should occur when the charge vector v can be written as $v = v' + v''$, where v', v'', and v are all mutually BPS. In view of (2.3), the condition for this is that

JHEP04(1999)017

$|v_+| = |v'_+| + |v''_+|$. Since $v = v_+$ in the near horizon geometry, this is $|v| = |v'_+| + |v''_+|$. In view of the triangle inequality, this is equivalent to saying that the projection of v' (or of v'') to V_+ is a multiple of v. In other words, the lattice Γ' generated by v and v' has a projection to V_+ that lies in a one-dimensional subspace, the space of multiples of v.

Assuming that this is so, the projection of Γ' to V_+ is one-dimensional, being generated by v, so Γ' has signature $(1,1)$. Given any primitive $v' \in \Gamma'$ and not a multiple of v, the requirement that the projection of v' to V_+ is a multiple of v puts four conditions on the near horizon moduli. (V_+ is five dimensional, so asking that a vector in V_+ be a multiple of v imposes four conditions.) If these conditions are imposed, then (as v and v' generate Γ') the projection of Γ' to V_+ consists of multiples of v. When this happens, every way of writing $v = v' + v''$ with $v', v'' \in \Gamma'$ and $(v')^2$, $(v'')^2 \geq 0$ will be a way of breaking our system into two BPS subsystems at no cost in energy. The ability to do this should give a singularity in the boundary conformal field theory.

Thus the loci in moduli space on which the CFT is expected to be singular are classified by signature $(1,1)$ sublattices $\Gamma' \subset \Gamma$ that contain v.[2] For each such Γ', a singularity is found by adjusting one hypermultiplet so that all projections of vectors in Γ' to V_+ become proportional. On this locus, the string can break up according to any decomposition $v = v' + v''$ with $v', v'' \in \Gamma'$ and $(v')^2$, $(v'')^2 \geq 0$. (For some Γ', there may exist no such v', v'', and then there is no singularity associated with Γ'.)

For example, let us classify all cases in which Γ' is unimodular, in other words in some basis its quadratic form is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.8}$$

To put it differently, Γ' is isomorphic to the unique even unimodular lattice $\Gamma^{1,1}$ of signature $(1,1)$. Such a Γ' has up to a duality transformation a unique embedding in $\Gamma^{5,n}$ (this is proved by noting that the orthocomplement of Γ' is even and unimodular and hence unique up to isomorphism). The transformation that puts Γ' in this standard form may rotate v into an arbitrary form, modulo the symmetries of Γ' plus the fact that v is primitive and $v^2 = 2N$. So in general we have $v = (Q_1, Q_5)$ for some relatively prime integers Q_1 and Q_5 with $Q_1 Q_5 = N$. By a symmetry of Γ' (namely $(Q_1, Q_5) \to (-Q_1, -Q_5)$), we can assume that Q_1 and Q_5 are both nonnegative. The only remaining symmetry of Γ' is $(Q_1, Q_5) \leftrightarrow (Q_5, Q_1)$. (For example, for $\mathbf{X} = \mathbf{T}^4$, this is a T-duality transformation on $\mathbf{X}$.) So, for given N, choices of a unimodular $(1,1)$ lattice Γ' containing v are classified by the unordered relatively prime pairs Q_1, Q_5 with $N = Q_1 Q_5$.

[2] Of course, we are only interested in the choice of Γ' up to the action of the duality group $\mathcal{H}$ that keeps v fixed. Equivalently, we want to choose the pair Γ' and v up to the action of the full duality group $\mathcal{K}$.

In other words, the singularities with unimodular Γ' are the small instanton and partial un-Higgsing singularities of $\mathrm{SU}(Q_5)$ gauge theory described in the introduction, for various values of Q_5. There are also singularities for which Γ' is not unimodular. (For example, if v describes the D1/D5 system while v' has threebrane charge as well as D1 and D5 charge, then v and v' can generate a lattice that is not unimodular.) In the present paper, our general analysis of the tube behavior via long strings in section 3 is valid for all of the singularities. But our more quantitative study in section 4 uses specifically the NS1/NS5 system (which is dual to D1/D5) and so is special to the case of unimodular Γ'.

We pause here to point out a subtlety that we have hidden in our exposition. Let $\Gamma_\perp$ be the sublattice of $\Gamma^{5,n}$ consisting of vectors perpendicular to v, and let $\mathcal{H}'$ be the automorphism group of $\Gamma_\perp$. $\mathcal{H}$ is a subgroup of $\mathcal{H}'$ (since any symmetry of $\Gamma^{5,n}$ that leaves v fixed must map $\Gamma_\perp$ to itself), and is actually a proper subgroup (any element of $\mathcal{H}'$ is a symmetry of the sublattice $v\mathbb{Z} \oplus \Gamma_\perp$ of $\Gamma^{5,n}$, but may not extend to a symmetry of $\Gamma^{5,n}$ itself). To describe the singularities associated with unimodular Γ', we have in the text classified, up to the action of $\mathcal{H}$, the embeddings of $\Gamma^{1,1}$ in $\Gamma^{5,n}$ that contain v. It would be tempting to reason as follows: Every such $\Gamma^{1,1}$ embedding contains a (unique up to sign) primitive vector $w \in \Gamma_\perp$, with $w^2 = -2N$. The choice of w determines the lattice $\Gamma^{1,1}$. So why not classify the $\Gamma^{1,1}$'s by classifying w up to the action of $\mathcal{H}'$? This reasoning actually gives the wrong result (w is unique up to the $\mathcal{H}'$ action, so one would conclude that the singularity depends only on N and not on Q_5), which is possible since $\mathcal{H}$, not $\mathcal{H}'$, is the symmetry group of the problem.

Examples. We conclude this section by briefly stating some examples. The properties we assert can all be verified in detail using formulas in [16].

For the D1/D5 system, the fixed scalars, which are entirely absent in the near horizon physics, are the volume of $\mathbf{X}$, the anti-self-dual part of the NS B-fields, and a linear combination of the RR zero-form and four-form. For example, in terms of the description of the D1/D5 system by a sigma model with instanton moduli space as the target, it is natural that the volume of $\mathbf{X}$ should drop out as the instanton equation on $\mathbf{X}$ is conformally invariant.

In order to see a singularity from separating the D1/D5 system into D1/D5 subsystems, four more parameters must be set to zero. They are the self-dual part of the NS B-fields and a linear combination of the RR zero-form and four-form. For example, it has been argued [18] that turning on the self-dual part of the B-field deforms the instanton equations to the equations for instantons in noncommutative Yang-Mills theory. This operation eliminates the small instanton singularity; it "blows up" the small instanton locus by, in one description, adding a constant to the ADHM equations. So this operation removes the singularity from breaking up the D1/D5 system into subsystems.

By an S-duality transformation, one can identify the corresponding statements for the NS1/NS5 system. The fixed scalars are the string coupling constant, the anti-self-dual part of the RR B-fields, and a linear combination of the RR zero-form and four-form. There is a singularity from breaking the NS1/NS5 system into similar subsystems if the remaining RR fields — the self-dual part of the two-form, and a linear combination of the zero-form and four-form — vanish.

In particular, in the study of the NS1/NS5 system by conventional conformal field theory methods [19]–[21], the RR fields are all assumed to vanish. Hence one is necessarily "sitting" on the singularity. As we explain in section 5, we believe that this is responsible for some apparent discrepancies between computations performed in the worldsheet and spacetime conformal field theories.

Comparison to symmetric product We conclude this section with a brief discussion of the much-discussed relation of the spacetime conformal field theory of the near horizon system with a given value of N to a sigma model with target space the symmetric product of N copies of $\mathbf{X}$, which we denote as $S^N\mathbf{X}$.

For reasons explained in [19, 16], the sigma model with $S^N\mathbf{X}$ as target very likely has a moduli space that agrees with (2.7). This alone suggests that the sigma model with target $S^N\mathbf{X}$ is the right model. Moreover, rigorous mathematical theorems[3] show that (for all N and certain charge vectors v with $v^2 = 2N$; the theorem has not been proved for all such vectors) the moduli space of instantons with suitably specified Chern classes is indeed birational to a symmetric product of N copies of $\mathbf{X}$. Further, it has been proved recently that any two birational compact hyper-Kähler manifolds are deformation equivalent. We believe that these facts mean that the D1/D5 system, for any N, is on the moduli space of the symmetric product.

Nevertheless, the relation between them is extremely subtle. The hyper-Kähler manifold $S^N\mathbf{X}$ has singularities where two points meet. Resolving the singularities and turning on the associated theta angle put us into a moduli space that parametrizes unknown objects that may have a rather complicated behavior. If it is true that this family of conformal field theories is the one we want, then these conformal field theories exhibit quite an assortment of singularities on many different loci in moduli space. We cannot rule out this hypothesis, and it seems plausible that it is true.

For the usual questions involving black holes in a macroscopic AdS model, one wants large Q_1 and Q_5, leading to very special small instanton singularities characteristic of the chosen charges. This may be described by a conformal field theory that can be connected to the symmetric product, but it cannot be described by the symmetric product itself – which indeed depends only on N and not on the separate choice of Q_1 and Q_5, and so cannot possibly yield the right singularities. We do not know where on the moduli space, if anywhere, the symmetric product point might be.

[3] For an expository survey of mathematical results cited in this paragraph with references, see [22].

It is tempting to think that the symmetric product point might be $Q_1 = N$, $Q_5 = 1$, with vanishing theta angles and RR fields. But as in general the D1/D5 system has singularities in codimension $4(Q_5 - 1)$, for $Q_5 = 1$ the system is "generically singular" whatever that means. This does not sound like the hallmark of the symmetric product point. We make a few more remarks on $Q_5 = 1$ in section 4.

3. Mechanism for the singularity

In this section, by enlarging upon comments by Maldacena, Michelson, and Strominger [12], we will give a microscopic mechanism for exhibiting the singularity of the Higgs branch. While the rest of the paper focusses only on the AdS_3 examples, in the present section we consider also AdS_{D+1} for all $D \geq 2$. ($D = 1$ has special properties explored in [12].)

As we explained in the introduction, the potential singularity of the Higgs branch arises when a brane is emitted from the system. Emission of the brane will lower the charges of the remaining system. For example, suppose that we are studying $\mathcal{N} = 4$ super Yang-Mills in four dimensions, with gauge group SU(N) (or SU(N) if one takes into account a "singleton" degree of freedom at infinity), via Type IIB on $AdS_5 \times \mathbf{S}^5$ with N units of five-form flux on $\mathbf{S}^5$. If the boundary of AdS_5 were flat, it would be possible to move on the Coulomb branch, Higgsing the SU(N) down to SU($N-1$) $\times$ SU(1) by giving an expectation value to a scalar field ϕ. In such a vacuum, at very short distances $\ll 1/|\phi|$, one sees a gauge group SU(N), but at longer distances this is reduced to SU($N-1$) $\times$ SU(1).

Let us try to translate this mechanism into AdS using the familiar IR-UV connection. The fact that the gauge group in the conformal field theory is SU(N) at very short distances means that very near the boundary of AdS_5, there are N units of five-form flux on $\mathbf{S}^5$. But at longer distances in the conformal field theory, or in other words farther from the boundary of AdS_5, there is only SU($N-1$), corresponding to $N-1$ units of five-form flux. But the flux in AdS_5 can only change in crossing a threebrane. So we assume that there is a very large region $W \subset AdS_5$, with a threebrane wrapped on ∂W, the boundary of W. Then the five-form flux is $N-1$ in W and N outside. If W is very large, then ∂W is roughly speaking "close to the boundary" of AdS_5. We will call a brane whose worldvolume is such a ∂W a "large brane," or, when it is a one-brane, a "long string." The SU($N-1$) $\times$ SU(1) low energy gauge symmetry of the Higgsed theory is in this situation interpreted as a SU(1) carried by the large threebrane plus the SU($N-1$) described by supergravity on $W \times \mathbf{S}^5$ with a flux of $N-1$.

In this particular example, because the boundary of AdS_5 has positive scalar curvature R, one expects the Coulomb branch to be suppressed because of an $R\phi^2$ interaction. We will see this behavior below as a divergence in the action or energy when the large threebrane approaches the boundary. We will also generalize the discussion to consider instead of AdS_5 a more general negatively curved Einstein five-manifold, whose boundary may have negative R. In that case, the threebrane action or energy will go to $-\infty$ when the threebrane approaches the boundary, reproducing the expected unstable behavior of the conformal field theory on a manifold of negative R.

Though we have framed this discussion for AdS_5, we can similarly probe the approach to the Coulomb branch in any of the AdS_{D+1} examples by considering the behavior of a large $D-1$-brane. Many of the properties are independent of D for $D > 2$, but special things happen for $D = 2$, where the $D-1$-brane is a string. As we will see, the effective theory of the large string is a Liouville theory. By studying it, we will be able to understand the singularity of the Higgs branch for the AdS_3 examples.

3.1 Analysis of the large brane

With the motivation that we have just explained, and following Maldacena, Michelson and Strominger [12], we study the dynamics of a large $D-1$ brane in AdS_{D+1}. We consider a large brane carrying charge q under the background antisymmetric tensor field. The brane is really moving on $AdS_{D+1}\times\mathbf{Y}$ for some $\mathbf{Y}$, but for the moment we can ignore the motion on $\mathbf{Y}$.

We let A denote the volume of ∂W, and we let V denote the volume of W. The brane action has two terms, one a positive multiple of A coming from the brane tension, and the other a negative multiple of V coming from the "Wess-Zumino coupling" of the brane to the background antisymmetric tensor field. In flat spacetime, for a sufficiently large brane one has $V \gg A$, so the action of a sufficiently large brane in the presence of a constant antisymmetric tensor field strength is negative. Hence a sufficiently large brane grows indefinitely. A constant antisymmetric tensor field in flat spacetime would relax to zero by nucleation of branes; for example, this mechanism leads to the periodicity of the θ angle in two dimensional QED. In AdS space, this energetics is more delicate because V is proportional to A. The BPS case is precisely the case that the leading volume and surface terms cancel. Much physics is contained in the subleading terms that do not cancel.

We start by writing down the metric of AdS space in the following form:

$$ds^2 = r_0^2(dr^2 + \sinh^2 r d\Omega^2)\,. \tag{3.1}$$

Here $d\Omega^2$ is the round metric on $\mathbf{S}^D$. The topology of the D dimensional worldvolume of the brane is $\mathbf{S}^D$ and it is located at $r(\Omega)$.

As a preliminary for finding the effective action of the brane we calculate V and A. We easily find that the volume enclosed by the brane is

$$V = r_0^{D+1}\int d^D\Omega \int_0^r dr' \sinh^D r' = \frac{r_0^{D+1}}{2^D}\int d^D\Omega \int_0^r dr' \left(e^{Dr'} - De^{(D-2)r'} + \mathcal{O}(e^{(D-4)r'})\right)$$

$$= \begin{cases} \frac{r_0^{D+1}}{2^D} \int d^D\Omega \left(\frac{1}{D} e^{Dr} - \frac{D}{D-2} e^{(D-2)r} + \mathcal{O}(e^{(D-4)r}) \right) & \text{for } D > 2\,; \\ \frac{r_0^3}{4} \int d^2\Omega \left(\frac{1}{2} e^{2r} - 2r + \mathcal{O}(e^{-2r}) \right) & \text{for } D = 2\,. \end{cases}$$

Here $d^D\Omega$ is the volume element of a unit sphere.

Similarly, the surface volume of the brane is

$$\begin{aligned} A &= r_0^D \int d^D\Omega \frac{1}{\sqrt{g}} \sqrt{\det_{\alpha\beta} \left(\sinh^2 r g_{\alpha\beta} + \partial_\alpha r \partial_\beta r \right)} \\ &= \frac{r_0^D}{2^D} \int d^D\Omega \left(e^{Dr} - D e^{(D-2)r} + 2e^{(D-2)r} (\partial r)^2 + \mathcal{O}(e^{(D-4)r}) \right), \end{aligned} \tag{3.2}$$

where $g_{\alpha\beta}$ is the round metric on the unit sphere.

In the above formulas, we can think of "r" as an effective field on the large brane. Before combining these formulas to compute the action of a large brane as a function of this field, we will put them in a more covariant form. This will make it clear how r transforms under Weyl transformations in the boundary theory, and why.

The boundary of AdS_{D+1} has a natural conformal structure but not a natural metric. Let $ds^2 = g_{ij} dx^i dx^j$ be an arbitrary metric on the boundary in its conformal class. Here the x^i, $i = 1, \ldots, D$ are an arbitrary set of local coordinates on the boundary. There is then (see Lemma 5.2 in [23]) a unique way to extend the x^i to coordinates on AdS_{D+1} near the boundary, adding an additional coordinate t that vanishes on the boundary, such that the metric in a neighborhood of the boundary is

$$ds^2 = \frac{r_0^2}{t^2} \left(dt^2 + \widehat{g}_{ij}(x,t) dx^i dx^j \right), \tag{3.3}$$

with

$$\widehat{g}_{ij}(x,0) = g_{ij}(x)\,. \tag{3.4}$$

Moreover, one can use the Einstein equations to determine the behavior of $\widehat{g}_{ij}$ near $t = 0$. One has

$$\widehat{g}_{ij}(x,t) = g_{ij}(x) - t^2 P_{ij} + \text{higher orders in } t\,, \tag{3.5}$$

where for $D > 2$

$$P_{ij} = \frac{2(D-1)R_{ij} - g_{ij}R}{2(D-1)(D-2)}\,, \tag{3.6}$$

which implies

$$g^{ij} P_{ij} = \frac{R}{2(D-1)}\,. \tag{3.7}$$

This last formula is the only property of P that we will need. For $D = 2$, (3.6) is no longer valid, but (3.7) is. (For $D = 2$, the Einstein equations do not determine the trace-free part of P in terms of local data near the boundary.)

In this formulation, we can see how t transforms under Weyl rescalings of the boundary metric. If we take $g_{ij} \to e^{2\omega} g_{ij}$, then t must be transformed to maintain the properties (3.3), (3.4). Clearly, this requires

$$t \to e^{\omega} t + \cdots \,, \tag{3.8}$$

where the ellipses are terms of higher order near $t = 0$. This means that t (if corrected by adding higher order terms to remove the ellipses) is a field of conformal dimension -1. The canonical dimension of a scalar field is $(D-2)/2$ for $D > 2$, so we should expect that a scalar field ϕ of canonical dimension will be $\phi \sim t^{-(D-2)/2}$ for $D > 2$. Instead, in classical field theory in $D = 2$, an ordinary scalar field is dimensionless. To make a field t of dimension -1 as a function of a two-dimensional real scalar field ϕ, ϕ must be a Liouville field (with a coupling ϕR to the worldsheet curvature R) and t must be written as a real exponential of ϕ.

Before computing V and A in the covariant approach, we set $t = 2e^{-r}$, so that the metric becomes

$$ds^2 = r_0^2 \left(dr^2 + \frac{e^{2r}}{4} g_{ij} dx^i dx^j - P_{ij} dx^i dx^j + \mathcal{O}(e^{-2r}) \right). \tag{3.9}$$

This will make the comparison with the formulas (3.2) and (3.2) more transparent. Also, as suggested in the last paragraph, we can write r in terms of a canonical scalar field ϕ for $D > 2$, or a Liouville field ϕ for $D = 2$, by

$$r = \begin{cases} \dfrac{2}{D-2} \log \phi + \dfrac{1}{(D-1)(D-2)} \phi^{-\frac{4}{D-2}} R & \text{for } D > 2 \\ \phi + e^{-2\phi} \phi R & \text{for } D = 2\,. \end{cases} \tag{3.10}$$

The leading terms in this formula, which correspond to $t = \text{const}\, \phi^{-2/(D-2)}$ for $D > 2$, and $t = \text{const}\, e^{-\phi}$ for $D = 2$, were explained in the last paragraph. The correction terms in (3.10) can presumably be calculated by computing the higher order terms in (3.8) and then seeing how t can be expressed in terms of a canonical scalar field for $D > 2$, or a Liouville field for $D = 2$. We will not be as systematic as this because the correction terms are unimportant for BPS branes, which are our main interest. We have merely determined the coefficients of the correction terms to make the formulas below conformally invariant even for the non-BPS case.

Using (3.9), we can compute that the volume enclosed by the brane, computed with an arbitrary metric on the boundary in its conformal class and using the associated r-function, is

$$V = \begin{cases} \dfrac{r_0^{D+1}}{2^D} \displaystyle\int d^D x \sqrt{g} \left(\frac{1}{D} e^{rD} - \frac{1}{(D-1)(D-2)} e^{(D-2)r} R + \mathcal{O}(e^{(D-4)r}) \right) & \text{for } D > 2\,; \\ \dfrac{r_0^3}{4} \displaystyle\int d^D x \sqrt{g} \left(\frac{1}{2} e^{2r} - rR + \mathcal{O}(e^{-2r}) \right) & \text{for } D = 2\,. \end{cases} \tag{3.11}$$

This reduces to (3.2) when the boundary is a unit sphere with a round metric, for which $R = D(D-1)$. In terms of the canonical scalar or Liouville field ϕ, we have

$$V = \begin{cases} \dfrac{r_0^{D+1}}{D2^D} \int d^D x \sqrt{g} \left(\phi^{\frac{2D}{D-2}} + \mathcal{O}(\phi^{\frac{2(D-4)}{D-2}}) \right) & \text{for } D > 2\,; \\ \dfrac{r_0^3}{8} \int d^D x \sqrt{g} \left(e^{2\phi} + \mathcal{O}(e^{-2\phi}) \right) & \text{for } D = 2\,. \end{cases} \tag{3.12}$$

Likewise, in the same generality, the surface volume of the boundary is

$$A = \frac{r_0^D}{2^D} \int d^D x \sqrt{g} \left(e^{Dr} - \frac{1}{D-1} e^{(D-2)r} R + 2e^{(D-2)r} (\partial r)^2 + \mathcal{O}(e^{(D-4)r}) \right). \tag{3.13}$$

In terms of ϕ, this is

$$A = \begin{cases} \dfrac{r_0^D}{2^D} \int d^D x \sqrt{g} \left(\phi^{\frac{2D}{D-2}} + \dfrac{8}{(D-2)^2} \left[(\partial\phi)^2 + \dfrac{D-2}{4(D-1)} \phi^2 R \right] + \mathcal{O}\left(\phi^{\frac{2(D-8)}{D-2}} \right) \right) & \text{for } D > 2 \\ \dfrac{r_0^2}{4} \int d^D x \sqrt{g} \left(e^{2\phi} + 2[(\partial\phi)^2 + \phi R] - R + \mathcal{O}(e^{-2\phi}) \right) & \text{for } D = 2\,. \end{cases} \tag{3.14}$$

One of the advantages of the covariant derivation that we have given is that these formulas are not restricted to branes near the boundary of AdS_{D+1}. One can replace AdS_{D+1} with an arbitrary Einstein manifold W of negative curvature and conformal boundary M. The same formulas hold, with the same derivation, for the area and Wess-Zumino coupling of a large brane that is near M, or, if M is not connected, near any component of M.

Now, let us combine these formulas and study the behavior of branes. The action of a brane that couples to the background antisymmetric tensor field with charge q receives two contributions. One term, arising from the tension of the brane T, is TA. The interaction of its charge with the background gauge fields leads to a term proportional to V. The whole action is

$$\begin{aligned} S =& T \left(A - \frac{qD}{r_0} V \right) = \\ =& \begin{cases} \dfrac{Tr_0^D}{2^D} \int \sqrt{g} \left((1-q) \phi^{\frac{2D}{D-2}} + \dfrac{8}{(D-2)^2} \left[(\partial\phi)^2 + \dfrac{D-2}{4(D-1)} \phi^2 R \right] + \mathcal{O}\left(\phi^{\frac{2(D-4)}{D-2}} \right) \right) & \text{for } D > 2; \\ \dfrac{Tr_0^2}{4} \int \sqrt{g} \left((1-q) e^{2\phi} + 2[(\partial\phi)^2 + \phi R] - R + \mathcal{O}(e^{-2\phi}) \right) & \text{for } D = 2. \end{cases} \end{aligned}$$

The brane approaches the boundary in the limit $\phi \to \infty$. For large ϕ, the dominant term is the first one, $(1-q)\phi^{2D/(D-2)}$ for $D > 2$ or $(1-q)e^{2\phi}$ for $D = 2$. For $q > 1$, the action goes to $-\infty$ for $\phi \to \infty$ and the system is unstable against the emission of branes. However, in a supersymmetric AdS vacuum, there will never be a brane with $q > 1$ as this violates a BPS bound. In a supersymmetric theory, we will have either BPS branes of $q = 1$ or non-BPS branes of $q < 1$.

If $q < 1$, the leading order term in (3.15) is a potential which tries to contract the brane. In this case, the effect of the tension of the brane is larger than the force due to the charge. The brane tends to contract and, if we are in AdS_{D+1}, eventually annihilates. If AdS_{D+1} is replaced by a more general Einstein manifold X, the non-BPS brane might conceivably contract to a stable minimum of the action.

For BPS branes, $q = 1$ and the effective action becomes

$$S = \begin{cases} \dfrac{Tr_0^D}{2^{D-3}(D-2)^2} \displaystyle\int \sqrt{g}\left((\partial\phi)^2 + \frac{D-2}{4(D-1)}\phi^2 R + \mathcal{O}\left(\phi^{\frac{2(D-4)}{D-2}}\right)\right) & \text{for } D > 2; \\ \dfrac{Tr_0^2}{2} \displaystyle\int \sqrt{g}\left((\partial\phi)^2 + \phi R - \frac{1}{2}R + \mathcal{O}(e^{-2\phi})\right) & \text{for } D = 2. \end{cases} \tag{3.15}$$

The BPS term has caused the "potential energy" term to cancel out, leaving an action that is for $D > 2$ the minimal conformally invariant kinetic energy of a scalar field, and for $D = 2$ is the minimal conformally invariant kinetic energy of a Liouville field.

Do BPS D-branes exist? For the usual AdS models with $D > 2$, they generally do. For example, there are BPS three-branes in the case of $AdS_5 \times \mathbf{S}^5$. The reason is that the usual $D > 2$ models are constructed from the near horizon geometry of a system of parallel branes of just one type. In such a model, a probe brane of the type used in building the vacuum is BPS. On the other hand, the AdS_3 or $D = 2$ examples are constructed from more than one type of brane, and then, as we have described in section 2, for generic values of the moduli there are no BPS probe branes.

Now, let us focus on what happens when BPS branes do exist. In string theory or M-theory on AdS_{D+1} (or more exactly $AdS_{D+1} \times Y$ for some Y) the boundary is $\mathbf{S}^D$, and its conformal structure is that of the round metric on a unit sphere. If S is evaluated with the round metric, which has $R > 0$, it is manifestly positive definite and divergent for $\phi \to \infty$. Since S is conformally invariant, this is true for any metric in the same conformal class. Thus, on AdS_{D+1}, there is no instability from emission of a large brane. This is in accord with [12], where it was shown that there is no such instability except for $D = 1$ (a case that we are not treating in the present paper).

On the other hand, suppose that we replace $AdS_{D+1} \times Y$ by $W \times Y$, where W is a more general Einstein manifold W of conformal boundary M. The conformal class of metrics on M may admit a representative with negative R. If so, S is not positive definite and the string theory on $W \times Y$ is unstable against emission of a large D-brane that approaches the boundary.

These results agree with field theory expectations in the important case that $D = 4$ and Y is, for example, $\mathbf{S}^5$. Then the boundary theory is $\mathrm{SU}(N)$ super Yang-Mills theory (or $\mathrm{SU}(N)$ if we include the singleton field on the boundary). If we formulate this theory on a four-manifold M in a conformally invariant fashion, and try to Higgs the $\mathrm{SU}(N)$ gauge group to $\mathrm{SU}(N-1) \times \mathrm{SU}(1)$ by giving an expectation value to a component ϕ of the scalar fields of the theory, then the conformally invariant kinetic energy for ϕ is precisely the functional S obtained above. Hence, at the field theory level, we expect an instability when M has negative scalar curvature (and more generally when the conformally invariant functional S is not positive semi-definite). In this way, string theory reproduces the instability that is evident in the field theory.

At this level, the results do not yet seem to distinguish $D = 2$ from $D > 2$. The conformal field theory of one of the $D = 2$ examples, just like those with $D > 2$, is stable if formulated on a boundary with $R > 0$, but not if $R < 0$. However, $D = 2$ is clearly a delicate case, since the growth of the action as a BPS brane approaches the boundary is much slower — linear in r for $D = 2$ and exponential for $D > 2$. This suggests that we should look at $D = 2$ more closely.

To understand what is special about $D = 2$, it helps to consider a Hamiltonian formalism and to consider the energy of a large brane rather than its action, as considered up to this point. Thus, we are now considering the large brane as a physical object, whereas so far, our branes (as Euclidean space objects) were really instantons. For this, we go to the Lorentz signature version of AdS_{D+1}, so that the boundary is now $\mathbb{R} \times \mathbf{S}^{D-1}$ (with $\mathbb{R}$ parametrizing the time direction) rather than $\mathbf{S}^D$. The formula for the large brane action S is still valid, and from it we can read off an effective Hamiltonian for the large brane. In particular, the energy has an $R\phi^2$ or $R\phi$ term for $D > 2$ or $D = 2$. The main point is that for $D > 2$, $\mathbb{R} \times \mathbf{S}^{D-1}$ has $R > 0$, and hence in the Hamiltonian formulation, the energy diverges as a brane approaches the boundary. But for $D = 2$, the boundary is $\mathbb{R} \times \mathbf{S}^1$ and has zero scalar curvature. Hence, the energy of the brane does not grow as the brane is stretched.

We can make a more precise statement using the fact that given the form of S, the effective theory on the brane is a Liouville theory (for a review see [24]). The normalizable states of Liouville the theory form a continuum above a certain positive threshold. In order to find the properties of the Liouville theory of the long string (3.15), it is convenient to rescale ϕ such that its kinetic term is canonical. Then, the coupling to the two dimensional curvature R leads to an improvement term in the stress tensor with coefficient

$$Q = \sqrt{4\pi T r_0^2} \tag{3.16}$$

leading to a threshold at

$$E_0 = \frac{Q^2}{4} = \pi T r_0^2 \,. \tag{3.17}$$

We conclude that a BPS brane in AdS_3 can be stretched to infinity with a finite cost in energy [12].

This finite non-zero energy is in accord with the fact that in the Euclidean version, a large brane approaching the boundary of AdS_3 has an action that diverges for $r \to \infty$, albeit slowly. Such a large brane is an "instanton." A finite action instanton describes a tunneling process to a state of degenerate energy. Since we are trying to get to a state of finite energy — with a large brane in real time — there cannot be a finite action instanton.

As promised in the introduction, we have found a dangerous region in the space-time conformal field theory of the D1/D5 system where it is described by an effective Liouville theory. We claim that, if the long string is a D1 brane, then the behavior for large ϕ corresponds to the small instanton singularity. The evidence for this claim is first that, as we explained at the outset, intuitively emission of a long string is a way to reduce the charges of the system. Second, the dangerous large ϕ region occurs precisely when the small instanton is BPS; when θ angles are turned on, as reviewed in section 2.2, to suppress the small instanton singularity, then the long string has $q < 1$ and the large ϕ region is suppressed by an exponential potential. Finally, we note the following important check of the identification of the small instanton singularity with the large ϕ region. The small instanton singularity of the D1/D5 system depends only on Q_5 and not Q_1 (since the singularity when an instanton is small is entirely independent of how many non-small instantons there are). This is entirely in accord with the fact that, if the long string is a D1 string, then the Liouville action $S(\phi)$ depends on Q_5 and not Q_1.

To be more precise about this, to describe the small instanton singularity of the D1/D5 system of charges (Q_1, Q_5), we use a long D-string plus a D1/D5 system of charges $(Q_1 - 1, Q_5)$. The action S actually describes the motion of the long D-string on AdS_3, and must be supplemented by additional terms describing its motion in other dimensions of $AdS_3 \times \mathbf{S}^3 \times \mathbf{X}$. In particular, the position of the long string on $\mathbf{X}$ corresponds, in the other description, to the position of the small instanton on $\mathbf{X}$.

Like the D1/D5 system, the other singularities described in the instroduction have an analogous tubelike description using other long BPS strings. Whenever there is a BPS string, the D1/D5 system (and its U-dual cousins) has a continuous spectrum at energies above (3.17), though the spectrum is discrete at energies smaller than this. The states in the continuum are states that contain a long string. States below threshold have wavefunctions that vanish exponentially in the part of phase space where there is a long string.

Likewise, the operators in the boundary conformal field theory have a discrete spectrum of conformal dimensions up to

$$\Delta_0 = \frac{E_0}{2} = \frac{1}{2}\pi T r_0^2\,. \tag{3.18}$$

The operators creating the long string and the fluctuations on it have continuous dimensions starting at Δ_0.

Conformal field theories with a continuous spectrum of dimensions are most familiar in cases when the target space is non-compact. The example of a single non-compact boson is well known. In this case the continuum is associated with the arbitrary momentum of the boson. Momentum conservation guarantees that

JHEP04(1999)017

in the operator product expansion of two operators there is only one value of the momentum, and therefore there is a discrete sum over the operators which appear there. When the target space is non-compact and is not translationally invariant, there is a continuum of operators in the operator product expansion. We now see that the D1/D5 system, which appears to have a compact target space $\mathcal{M}$ (the moduli space of $\mathrm{SU}(Q_5)$ instantons with instanton number Q_1), also has a continuous spectrum, albeit above a gap. This is possible because $\mathcal{M}$ has singularities. The description of the conformal field theory as a sigma model is not very useful near singularities of $\mathcal{M}$. The good description of the behavior near the singularities is in terms of a long string plus a residual system of lower charges.

For the D1/D5 system, the long string is a D-object; it can be any collection of D1, D3 and D5 branes which forms a BPS string. Therefore, from the point of view of a conformal field theory description of the D1/D5 system (such as the one in [25]), the long string effect is non-perturbative. The same system, in a different region of its parameter space, has a more natural description as an NS1/NS5 system, built from NS5 branes and weakly coupled fundamental strings. In this region, the lightest string which can escape to infinity is a fundamental string. Hence our instanton is in this case a genus zero worldsheet instanton. It contributes at string tree level, but its contribution is non-perturbative in $\alpha' = \frac{1}{2\pi T}$.

The string coupling constant The boundary conformal field theory is a conformal field theory in which the string coupling constant $\lambda_{eff}(\phi)$ blows up as $\phi \to \infty$, that is, as the long string goes to infinity. This is because of the ϕR coupling. The divergence of λ is actually the reason that the action for the instantonic long string — wrapped around the boundary of Euclidean AdS_3 — diverges as ϕ goes to infinity. (The instanton amplitude is proportional to $\exp(-S) \sim \exp(-\sqrt{4\pi T r_0^2}\phi)$, which we can interpret as $1/\lambda_{eff}^2(\phi)$.) The divergence of $\lambda_{eff}(\phi)$ also causes the partition function of the boundary conformal field theory to be divergent for $\phi \to \infty$ if formulated on a Riemann surface of genus > 1. We have already noted above this consequence of the $R\phi$ coupling for the case that $R < 0$.

If we formulate the boundary CFT in genus zero (for instance on the boundary of AdS_3), then because $R > 0$, the partition function converges. However, one would suspect that the natural operators of the boundary CFT will have an exponential dependence on ϕ, and that some correlation functions will diverge in integrating over ϕ. Thus, some correlation functions of the boundary CFT of the D1/D5 system are likely to receive divergent contributions from the small instanton singularity, as well as singularities from partial un-Higgsing. Of course, as reviewed in section 2, these divergent contributions can be avoided if one turns on suitable theta angles and RR fields so that there are no BPS-saturated strings. In any event, determining the ϕ dependence of operators and showing that there really are divergent contributions to genus zero correlation functions is beyond the scope of the present paper.

JHEP04(1999)017

For some applications of conformal field theories of this kind, the divergence of the effective string coupling for $\phi \to \infty$ is very important. For example, if we set $Q_5 = 2$, then the small instanton singularity is in codimension four and looks like $\mathbb{R}^4/\mathbb{Z}_2$. Specializing to this case, the analysis shows that in terms of the right variable (which is the long string position) the effective string coupling in $\mathbb{R}^4/\mathbb{Z}_2$ conformal field theory (with zero theta angle) goes to infinity as one approaches the singularity. This makes it possible for Type IIA string theory on $\mathbb{R}^4/\mathbb{Z}_2$ to have nonperturbative behavior (enhanced gauge symmetry) no matter how small the bare string coupling constant may be.

Change in central charge Because of the ϕR term in the action (3.15), the Liouville theory of the long string has central charge

$$c = 3Q^2 = 12\pi T r_0^2\,. \tag{3.19}$$

We can interpret the long string as carrying away that amount of c out of the full conformal field theory of the D1/D5 system.

For the D1/D5 system, using $Tr_0^2 = \frac{Q_5}{2\pi}$, the threshold (3.18) and the central charge (3.19) become

$$\Delta_0 = \frac{Q_5}{4}c = 6Q_5\,. \tag{3.20}$$

The boundary conformal field theory of the full system is known to have $c_{full} = 6Q_1Q_5$. When a string is emitted, the remaining system has charges $(Q_1 - 1, Q_5)$, so its central charge is reduced by $6Q_5$. This coincides with the central charge of the long string itself, so we see that the string whose emission reduces Q_1 by one carries with it the remaining value of c. This is compatible with the claim that in a certain region of its phase space, the (Q_1, Q_5) system behaves like a long string plus a $(Q_1 - 1, Q_5)$ system.

The formulas in (3.20) are, however, only semiclassical approximations. A more precise derivation will be presented in section 4.

4. Exact analysis of the long string

The preceding analysis of the central charge and the gap is valid only for $Q_1, Q_5 \gg 1$, where the the long string does not affect the background geometry significantly. We now present a more complete analysis, for the weakly coupled NS5/NS1 system. It is valid for $Q_1 \gg 1$ but without assuming that Q_5 is large.

We will explain several approaches. To start with, we use the RNS formalism as in [19]–[21]. The system has short strings (ordinary perturbative excitations) and the long strings that we have been considering. Using a coordinate system with the metric

$$ds^2 = dx^2 + e^{2x} d\gamma d\bar{\gamma}\,, \tag{4.1}$$

the worldsheet lagrangean is

$$\int \left(\partial x\bar{\partial}x + e^{2x}\bar{\partial}\gamma\partial\bar{\gamma}\right) d^2z\,. \tag{4.2}$$

By introducing auxiliary fields β, $\bar{\beta}$, one can write an equivalent lagrangean

$$\int \left(\partial x\bar{\partial}x + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - e^{-2x}\bar{\beta}\beta\right) d^2z\,. \tag{4.3}$$

A long string can be described by the solution of the worldsheet equations of motion

$$\begin{aligned} &x = x_0 \gg 1 \\ &\gamma(z,\bar{z}) = z \;\; \bar{\gamma}(z,\bar{z}) = \bar{z}\,. \end{aligned} \tag{4.4}$$

Since we take $x_0 \gg 1$, we can treat the $e^{-2x}\bar{\beta}\beta$ term in (4.3) as a small perturbation, and use the free field representation of SL(2) conformal field theory as in [19]. Furthermore, the calculation of the central charge of the space-time Virasoro algebra in [19] applies directly to the long string rather than to the whole system. In particular, the value of the central charge found there, namely

$$c = 6Q_5 \oint dz \frac{\partial\gamma}{\gamma} = 6Q_5\,, \tag{4.5}$$

coincides with our semiclassical result (3.20). More generally, if γ in (4.4) is replaced by $\gamma = z^m$, which corresponds to m coincident long strings, (4.5) becomes [19]

$$c = 6Q_5 m\,. \tag{4.6}$$

In the rest of this paper we will be mostly concerned with a single long string; i.e. $m = 1$.

This calculation is to be contrasted with the calculation of the central charge of the full theory including the short strings [20, 21], which receives contributions from disconnected diagrams. For more details about these two calculations and the relation between them see [21].

We do not, however, know how to compute the value of the threshold from this point of view. We will therefore present two additional calculations which determine both the central charge of the long string and the threshold. The first computation is based on the covariant RNS description with ghosts, and the second on a physical gauge. (The first of the two calculations is somewhat intuitive and needs to be put on a firmer footing than we will achieve.)

We start by considering the bosonic string on $AdS_3 \times \mathbf{M}$ ($\mathbf{M}$ might be, for instance $\mathbf{S}^3 \times \mathbf{T}^{20}$). In the covariant formalism the worldsheet stress tensor is

$$T_{worldsheet} = T_{\mathrm{SL}(2)} + T_{\mathbf{M}} + T_{ghosts}\,, \tag{4.7}$$

where $T_{\mathrm{SL}(2)}$ is constructed out of a bosonic $\mathrm{SL}(2)_k$ WZW theory. $T_{\mathbf{M}}$ is the stress tensor of the conformal field theory on $\mathbf{M}$ and T_{ghosts} is constructed out of the b, c ghosts. The central charges of these stress tensors are

$$\begin{aligned} c_{\mathrm{SL}(2)} &= \frac{3k}{k-2} \\ c_{\mathbf{M}} &= 26 - \frac{3k}{k-2} \\ c_{ghosts} &= -26\,. \end{aligned} \tag{4.8}$$

Note that the total central charge $c_{worldsheet} = c_{\mathrm{SL}(2)} + c_{\mathbf{M}} + c_{ghosts}$ vanishes.

In the Wakimoto representation, using the fields in (4.3), the three SL(2) currents of the bosonic level k WZW theory are

$$\begin{aligned} J^- &= \beta \\ J^3 &= \beta\gamma + \frac{1}{2}\sqrt{2(k-2)}\partial x \\ J^+ &= \beta\gamma^2 + \sqrt{2(k-2)}\gamma\partial x + k\partial\gamma\,. \end{aligned} \tag{4.9}$$

Evaluating them on the classical solution of the long string we find

$$\begin{aligned} J^- &= 0 \\ J^3 &= 0 \\ J^+ &= k\,. \end{aligned} \tag{4.10}$$

Therefore, it is natural to impose a lightcone-like gauge[4] $J^+ = k$. More precisely, we study the BRST cohomology of our string after expanding the fields around this classical configuration. To compute this cohomology, we imitate the proof of the no ghost theorem for the bosonic string in flat space, as presented in [27]. Because γ only appears in the lagrangean via a term proportional to $\bar\partial\gamma$, there is a symmetry $\gamma \to \gamma + z$. We define an operator N (analogous to N^{lc}, introduced in [27], eqn. (4.4.8)) that acts as $\oint J^3$ on all modes of β, γ, and ϕ, except the mode $\gamma \sim z$ on which it acts trivially. Because of that one mode, the operator N does *not* commute with Q_{BRST}. The term in Q_{BRST} with the most negative N charge is $\widetilde{Q}_{BRST} = \oint cJ^-$. The equation $Q_{BRST}^2 = 0$ reduces, for the term of most negative N charge, to $\widetilde{Q}_{BRST}^2 = 0$. The same argument as in the usual proof of the no ghost theorem in flat space shows that the cohomology of Q_{BRST} is the same as that of $\widetilde{Q}_{BRST}$. Also, as in flat space, taking the $\widetilde{Q}_{BRST}$ cohomology effectively removes from the Hilbert space the fields β and γ along with the conformal ghosts b and c. The cohomology consists of all states built by acting on the ground state with the other fields.[5]

[4] Related ideas were explored in [26].

[5] A very similar no ghost argument for the $\mathrm{SL}(2,\mathbb{R})$ WZW model, using a free field representation, can be found in [28]. For another approach to this model and its no ghost theorem, see [29].

The procedure just sketched is similar to that of the Hamiltonian reduction of SL(2) as in [30], but the problem we are studying and therefore also the details of the construction are somewhat different. We are computing the physical state spectrum for a bosonic string on $AdS_3 \times \mathbf{M}$, so in particular we have the bosonic string b, c ghosts of spins -2 and 1 together with additional variables, while in Hamiltonian reduction a different problem is solved, so the $\mathbf{M}$ degrees of freedom are absent, and different ghosts are used. Our gauge fixing condition is therefore also somewhat different, though the computation of the cohomology ends up being similar.

Now we want to find the spacetime stress tensor. It acts on the BRST cohomology, which we have identified as the cohomology of the operator $\oint cJ^-$. In this representation, the spacetime stress tensor must be an operator that commutes with $\oint cJ^-$. For this to be so, J^- must have conformal dimension 2 relative to the spacetime stress tensor (though its dimension with respect to the worldsheet stress tensor is 1). Also, the current J^+, which is expanded around a constant, should have conformal dimension zero. These dimensions are obtained by twisting the stress tensor (4.7) to

$$T_{total} = T_{worldsheet} + \partial J^3 \,. \tag{4.11}$$

We interpret T_{total} as the space-time stress tensor of the theory on the long string. Computing the central charge of (4.11) using $J^3(z)J^3(w) \sim -\frac{k/2}{(z-w)^2}$ we find

$$c_{total} = 6k \,. \tag{4.12}$$

The physical degrees of freedom living on the long string include the modes of the sigma model on $\mathbf{M}$ describing the location of the string in $\mathbf{M}$, and the x boson which we now identify as the ϕ field of section 3. The central charge (4.12) is obtained because the boson ϕ becomes a Liouville field with an improvement term Q. In order to find the improvement term we equate the two expressions for the central charge

$$26 - \frac{3k}{k-2} + 1 + 3Q^2 = 6k \,, \tag{4.13}$$

and find

$$Q = (k-3)\sqrt{\frac{2}{k-2}} \,. \tag{4.14}$$

Because of this improvement term the system has a gap

$$\Delta_0 = \frac{Q^2}{8} = \frac{(k-3)^2}{4(k-2)} \,. \tag{4.15}$$

The two computations we have so far described give the same formula for the central charge in the spacetime conformal field theory, but give seemingly very different formulas for the spacetime stress tensor. How are they related? This question can be partly answered as follows. In the classical approximation of large k we ignore the ghosts and set the worldsheet stress tensor (4.7) to zero

$$0 = \frac{-(J^3)^2 + J^+J^-}{k} + T_{\mathbf{M}} \tag{4.16}$$

(we neglect the shift of k by 2). We substitute $J^+ = k$ and solve for J^-

$$J^- = \frac{(J^3)^2}{k} - T_{\mathbf{M}}\,. \tag{4.17}$$

Equation (2.37) in [19] gives the spacetime Virasoro generators

$$L_n = \oint dz \left[nJ^-\gamma^{n+1} - (n+1)J^3\gamma^n\right] = \oint dz \left[-J^-\gamma^{n+1} - \frac{1}{2}(n+1)\sqrt{2k}\partial x\gamma^n\right], \tag{4.18}$$

where we used the large k limit of (4.9). Substituting (4.17) in (4.18) and ignoring terms which vanish for large k we find

$$L_n = \oint dz \left[\left(T_{\mathbf{M}} - \frac{1}{2}(\partial x)^2\right)\gamma^{n+1} - \frac{1}{2}(n+1)\sqrt{2k}\partial x\gamma^n\right]. \tag{4.19}$$

Using our gauge choice $\gamma = z$

$$L_n = \oint dz \left[T_{\mathbf{M}} - \frac{1}{2}(\partial x)^2 + \frac{1}{2}\sqrt{2k}\partial^2 x\right] z^{n+1}\,. \tag{4.20}$$

Therefore, the spacetime stress tensor is

$$T_{spacetime} = T_{\mathbf{M}} - \frac{1}{2}(\partial x)^2 + \frac{1}{2}\sqrt{2k}\partial^2 x\,, \tag{4.21}$$

whose central charge for large k is $c = 6k$ in agreement with the exact answers. Note also that the boson x acquires an improvement term as above.

We now extend this analysis to superstrings on $AdS_3 \times \mathbf{S}^3 \times \mathbf{X}$ in the RNS formalism. The worldsheet stress tensor is

$$T_{worldsheet} = T_{\mathrm{SL}(2)} + T_{\mathrm{SU}(2)} + T_{\mathbf{X}} + T_{ghosts}\,. \tag{4.22}$$

$T_{\mathrm{SL}(2)}$ is constructed out of a bosonic $\mathrm{SL}(2)_{Q_5+2}$ WZW theory and three free fermions $\psi^{\pm}$ and ψ^3. $T_{\mathrm{SU}(2)}$ is constructed out of a bosonic $\mathrm{SU}(2)_{Q_5-2}$ WZW and three free fermions, χ^a. $T_{\mathbf{X}}$ is the stress tensor of the superconformal field theory on $\mathbf{X}$. T_{ghosts} is constructed out of the b, c and $\tilde{b}, \tilde{c}$ ghosts. The central charges of these stress tensors are

$$\begin{aligned}
c_{\mathrm{SL}(2)} &= \frac{3(Q_5+2)}{Q_5} + \frac{3}{2} \\
c_{\mathrm{SU}(2)} &= \frac{3(Q_5-2)}{Q_5} + \frac{3}{2} \\
c_{\mathbf{X}} &= 6 \\
c_{ghosts} &= -15.
\end{aligned} \tag{4.23}$$

As in the bosonic problem the total central charge $c_{worldsheet} = c_{\mathrm{SL}(2)} + c_{\mathrm{SU}(2)} + c_{\mathbf{X}} + c_{ghosts}$ vanishes.

We can now fix a lightcone gauge $J^+ = Q_5 + 2$. The BRST charge then has a term proportional to $\widetilde{Q}_{BRST} = \oint c(J^- + \psi^3\psi^-) + \tilde{c}\psi^-$, and higher order terms. Again, the BRST cohomology of the full BRST charge is the same as that of $\widetilde{Q}_{BRST}$. As above, the spacetime stress tensor is obtained by twisting the stress tensor (4.22)

$$T_{total} = T_{worldsheet} + \partial(J^3 + \psi^+\psi^-)\,, \tag{4.24}$$

such that J^+, J^-, ψ^+ and ψ^- have conformal dimensions 0, 2, $-1/2$, and $3/2$ respectively. As in the bosonic problem, the twisting changes the central charge of the stress tensor (4.24) to

$$c_{total} = 6Q_5 \tag{4.25}$$

in agreement with the exact result above.

The physical degrees of freedom living on the long string include the bosonic $\mathrm{SU}(2)_{Q_5-2}$, the three free fermions χ^a, the modes of the sigma model on $\mathbf{X}$, the ϕ boson and its superpartner which we denote by ψ. The other modes on AdS and the ghosts are effectively removed by considering the cohomology. The central charge (4.25) is obtained because the boson ϕ becomes a Liouville field with an improvement term $Q = (Q_5 - 1)\sqrt{2/Q_5}$. Because of this improvement term the system has a gap

$$\Delta_0 = \frac{Q^2}{8} = \frac{(Q_5-1)^2}{4Q_5}\,. \tag{4.26}$$

This result is consistent with the semiclassical answer $Q_5/4$ we found above.

Our identification of the degrees of freedom on the long string and the fact that this theory is essentially a free CFT depend on having a single long string; i.e. $m = 1$ in (4.6). For $m > 1$ coincident long strings, the collective coordinate ϕ describes their center of mass, but there are also other degrees of freedom corresponding to the separation between them. These degrees of freedom lead to $c = 6Q_5 m$ rather than $c = 6Q_5$. As is common in D-branes, the center of mass theory can be free but the remaining degrees of freedom are interacting.

Spacetime supersymmetry Finally, we will present a description of the long string that is more precise than the above and exhibits spacetime supersymmetry. For this, we will use a sort of unitary gauge description in terms of physical degrees of freedom only, an approach we followed already in section 3. The reason for using unitary gauge is that to see spacetime supersymmetry, one would like to use a Green-Schwarz type description, but the Green-Schwarz string is difficult to quantize covariantly.[6]

[6]For a recent attempt to derive a somewhat similar description of AdS_3 models from the Green-Schwarz string, see [31].

JHEP04(1999)017

For simplicity, consider the case $\mathbf{X} = \mathbf{T}^4$ (the extension to $\mathbf{K3}$ is straightforward). We start with an RNS description and then pass to Green-Schwarz variables by introducing spin fields. The sigma model on $\mathbf{X}$ is described by four free bosons x^i and four free fermions ψ^i. Motion on $\mathbf{S}^3$ is described by an SU(2) current algebra. In a gauge $\gamma = z$, the long string motion on AdS_3 is described by a Liouville field ϕ introduced in section 3. In the RNS description, two worldsheet fermions that are superpartners of $\gamma, \bar{\gamma}$ can be set to zero by a gauge choice. In this unitary gauge, all ghosts decouple. So the description is by x^i, ϕ, and the SU(2) current algebra, plus eight fermion partners.

To go to a Green-Schwarz description with manifest spacetime supersymmetry, one replaces the eight RNS fermions by their eight spin fields, which have dimension 1/2 and are free fermions. In terms of these eight fermion fields along with x^i, ϕ, and the SU(2) currents j^a, we want to construct an $\mathcal{N} = 4$ superconformal algebra which we will interpret as the spacetime superconformal algebra.

The x^i together with four free fermions make an $\mathcal{N} = 4$ hypermultiplet, with $c = 6$. So the essential point is to construct from ϕ, the SU(2) currents j^a at level $Q_5 - 2$, and four free fermions S^μ, an $\mathcal{N} = 4$ algebra that, roughly, describes the long string motion on $AdS_3 \times \mathbf{S}^3$. The nontrivial operator product expansions are

$$\begin{aligned} S^\mu(z)S^\nu(w) &\sim -\frac{\delta^{\mu\nu}}{z-w} \\ \partial\phi(z)\partial\phi(w) &\sim -\frac{1}{(z-w)^2} \\ j^a(z)j^b(w) &\sim -\frac{\delta^{ab}(Q_5-2)/2}{(z-w)^2} + \frac{\epsilon^{abc}j^c}{(z-w)}\,. \end{aligned} \tag{4.27}$$

We use the 't Hooft η symbol

$$\eta^a_{\mu\nu} = \frac{1}{2}(\delta_{a\mu}\delta_{0\nu} - \delta_{a\nu}\delta_{0\mu} + \epsilon_{a\mu\nu}) \tag{4.28}$$

to express the $\mathcal{N} = 4$ generators

$$\begin{aligned} T &= -\frac{1}{2}\partial S^\mu S^\mu - \frac{j^a j^a}{Q_5} - \frac{1}{2}\partial\phi\partial\phi + \frac{\sqrt{2}(Q_5-1)}{2\sqrt{Q_5}}\partial^2\phi \\ J^a &= j^a + \frac{1}{2}\eta^a_{\mu\nu}S^\mu S^\nu \\ G^\mu &= \frac{\sqrt{2}}{2}\partial\phi S^\mu - \frac{2}{\sqrt{Q_5}}\eta^a_{\mu\nu}j^a S^\nu + \frac{1}{6\sqrt{Q_5}}\epsilon_{\mu\nu\rho\sigma}S^\nu S^\rho S^\sigma - \frac{Q_5-1}{\sqrt{Q_5}}\partial S^\mu\,. \end{aligned} \tag{4.29}$$

A straightforward calculation shows that they satisfy the $\mathcal{N} = 4$ algebra with $c = 6(Q_5 - 1)$

$$J^a(z)J^b(w) \sim -\frac{\delta^{ab}(Q_5-1)/2}{(z-w)^2} + \frac{\epsilon^{abc}J^c}{(z-w)}$$

$$\begin{aligned}
T(z)J^a(w) &\sim \frac{J^a}{(z-w)^2} + \frac{\partial J^a}{(z-w)} \\
T(z)T(w) &\sim \frac{3(Q_5-1)}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial T}{z-w} \\
T(z)G^\mu(w) &\sim \frac{\frac{3}{2}G^\mu}{(z-w)^2} + \frac{\partial G^\mu}{(z-w)} \\
J^a(z)G^\mu(w) &\sim \frac{\eta^a_{\mu\nu}}{z-w}G^\nu \\
G^\mu(z)G^\nu(w) &\sim \frac{\delta^{\mu\nu}2(Q_5-1)}{(z-w)^3} - \frac{4}{(z-w)^2}\eta^a_{\mu\nu}J^a + \frac{1}{z-w}\left(\delta^{\mu\nu}T - 2\eta^a_{\mu\nu}\partial J^a\right). \qquad (4.30)
\end{aligned}$$

Together with the $\mathcal{N}=4$ algebra of the $\mathbf{T}^4$, which has $c=6$, we have the expected result of $\mathcal{N}=4$ with $c=6Q_5$.

If we remove the improvement terms (last terms) in T and G^μ and add the free fermions S^μ, the SU(1) current $\partial\phi$ and the $\mathrm{SU}(2)_1$ currents $\widetilde{J}^a = \frac{1}{2}\bar{\eta}^a_{\mu\nu}S^\mu S^\nu$, we find a realization of the large $\mathcal{N}=4$ algebra [32]–[35]. This algebra has two different ordinary $\mathcal{N}=4$ subalgebras: the one above with $c=6(Q_5-1)$, and another one including $\widetilde{J}^a$ but without J^a with $c=6$. The other $\mathcal{N}=4$ algebra appeared in the study of string propagation on solitons [36].

These two constructions are also important in the closely related gauge theory on the onebranes. The algebra with $c=6$ appears in the tube of the Coulomb branch [6]. The algebra (4.29) with $c=6(Q_5-1)$ appears along the Higgs branch of the same system. Note that as in [4], they have different R symmetries and different central charges.

The chiral operators of the $\mathcal{N}=4$ algebra of the long string theory (4.30) are easily found to be

$$\mathcal{O}_j = e^{j\sqrt{\frac{2}{Q_5}}\phi}V_j\,, \qquad j = 0, \frac{1}{2}, \ldots, \frac{Q_5-2}{2}\,, \qquad (4.31)$$

where V_j are the spin j operators in the bosonic $\mathrm{SU}(2)_{Q_5-2}$ WZW theory. The exponent in the first factor is determined by imposing

$$\Delta(\mathcal{O}_j) = j\,, \qquad (4.32)$$

or by demanding that there is no isospin $j+1/2$ operator in the simple pole in the operator product expansion of $G^\mu\mathcal{O}_j$.

The chiral operators $\mathcal{O}_{\frac{1}{2}}$ lead to the following four desendants of conformal dimension $(1,1)$ (the other descendants at this level are null since $\mathcal{O}_{\frac{1}{2}}$ is chiral):

$$\delta L = \left\{\bar{G}, \left[G, \mathcal{O}_{\frac{1}{2}}\right]\right\}. \qquad (4.33)$$

They are invariant under the left moving and right moving $\mathrm{SU}(2)_{Q_5-1}$ current algebras in the $(4,4)$ symmetry. As is standard in $\mathcal{N}=4$ superconformal field theories,

JHEP04(1999)017

such operators are truly marginal and preserve the $(4,4)$ superconformal symmetry. With these operators added to the lagrangean we find $\mathcal{N}=4$ Liouville.

As we explained in section 3, if one perturbs the system so that the long string is not BPS, such a Liouville potential is generated. Let us check that the operators (4.33) have the expected quantum numbers. The operators (4.33) transform as $(\mathbf{2},\mathbf{2})$ under the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ outer automorphism of the $(4,4)$ superconformal algebra. In the theory of the Higgs branch — understood in terms of a small instanton on $\mathbb{R}^4$ — the outer automorphism group acts by rotations of $\mathbb{R}^4$. Since the noncompact bosons describing motion on $\mathbb{R}^4$ cannot be decomposed as sums of left and right-movers, only a diagonal subgroup $\mathrm{SU}(2)_D$ of the outer automorphism group is a symmetry. (If one embeds the small instanton in $\mathbf{X}=\mathbf{T}^4$ or $\mathbf{K3}$, this explicitly breaks $\mathrm{SU}(2)_D$. But for a sufficiently small instanton, the $\mathrm{SU}(2)_D$ symmetry is a good approximation, as the small instanton singularity does not "see" the compactification of $\mathbb{R}^4$ to $\mathbf{X}$.) Under $\mathrm{SU}(2)_D$, the four operators (4.33) transform as $\mathbf{1}\oplus\mathbf{3}$.

Now we compare this to the D1/D5 system. As reviewed in section 2.2, this system can be deformed away from a singular point by turning on a θ angle or FI terms. These transform as $\mathbf{1}\oplus\mathbf{3}$ of $\mathrm{SU}(2)_D$, just the quantum numbers of the $(1,1)$ operators (4.33).

Liouville theory with improvement term proportional to Q has operators $e^{\alpha\phi}$ with dimensions $\Delta_\alpha=-\frac{1}{2}\alpha(\alpha-Q)$. The values of α are constrained by [24]

$$\begin{array}{ll} \alpha\leq\frac{Q}{2} & \text{for } Q>0 \\ \alpha\geq-\frac{Q}{2} & \text{for } Q<0\,. \end{array} \tag{4.34}$$

The string coupling depends on ϕ according to

$$g(\phi)=e^{Q\phi/2}\,, \tag{4.35}$$

and therefore the coupling is strong as $\phi\to+\infty$ ($\phi\to-\infty$) for Q positive (negative). The wave function associated with the operator $e^{\alpha\phi}$ is

$$\psi(\phi)=\frac{1}{g(\phi)}e^{\alpha\phi}=e^{(\alpha-\frac{Q}{2})\phi}\,. \tag{4.36}$$

The condition (4.34) can be interpreted as the condition that the wave function diverges at the weak coupling end [24].

In our problem we have two Liouville systems. The first is the worldsheet theory in the AdS_3 background. For large ϕ, it becomes a free theory with an improvement term with $Q_C=-\sqrt{2/Q_5}$. This value is obtained by looking at the shift term of the free Wakimoto description of our system [19]. Alternatively, it can also be obtained by analyzing the theory on the Coulomb branch and its tube (hence the subscript C) [36]. This system describes the short strings. The second Liouville system is on the long string and corresponds to the tube of the Higgs branch. From that point of view we have seen that large ϕ corresponds to strong coupling. The Liouville system of the long strings has $Q_H = (Q_5 - 1)\sqrt{2/Q_5}$. Q_C and Q_H differ in sign and in the absolute value. These two facts follow from the two $N = 4$ superconformal subalgebras of the large $N = 4$ [33].

It is natural to identify ϕ of the two tubes. The string coupling for the short strings is large at one end and the string coupling for the long strings is large at the other end:

$$\begin{aligned} g_C(\phi) &= e^{Q_c\phi/2} = \exp\left(-\frac{1}{\sqrt{2Q_5}}\phi\right) \\ g_H(\phi) &= e^{Q_H\phi/2} = \exp\left(\frac{Q_5-1}{\sqrt{2Q_5}}\phi\right). \end{aligned} \tag{4.37}$$

The vertex operators $e^{\alpha\phi}$ can act either on short strings or on long strings. Therefore, they should make sense in the two Liouville systems and hence[7]

$$-\frac{1}{\sqrt{2Q_5}} \le \alpha \le \frac{Q_5-1}{\sqrt{2Q_5}}\,. \tag{4.38}$$

This guarantees that their wave functions diverge at the weak coupling end both for short strings and for long strings.

The fact that the wave function $\psi(\phi)$ of a local vertex operator $\mathcal{O}$ diverges at large ϕ for a small string state but vanishes exponentially at large ϕ for a long string seems strange at first, but is not so hard to understand intuitively. It means roughly that $\mathcal{O}$ is likely to create a small string near the boundary of AdS_3, but is very unlikely to create a long string wrapped around the boundary.

It is reassuring to see that the chiral operators (4.31) satisfy (4.38). We will have more remarks about this bound in section 5.

The chiral operators (4.31) can be compared with their counterparts in the first quantized description of [19] (equation (4.11)):

$$\int d^2z e^{-\varphi-\bar\varphi}\left(\psi^3 - \frac{1}{2}\gamma\psi^- - \frac{1}{2}\gamma^{-1}\psi^+\right)\left(\bar\psi^3 - \frac{1}{2}\bar\gamma\bar\psi^- - \frac{1}{2}\bar\gamma^{-1}\bar\psi^+\right)e^{j\sqrt{\frac{2}{Q_5}}\phi}\gamma^{j+m}\bar\gamma^{j+\bar m}V_j\,, \tag{4.39}$$

where φ and $\bar\varphi$ are the bosonized ghosts. This expression is in the minus one ghost picture. In the zero picture the exponential of the ghosts disappears and the factors with the fermions are replaced by the SL(2) currents and some higher order terms. We follow the light-cone like gauge fixing (4.10) and keep only the term proportional to J^+, which is a constant in our gauge. Using $\gamma = z$ we recognize (4.39) as the $(m, \bar m)$ mode of the local operator $e^{j\sqrt{2/Q_5}\phi}V_j$. This is the chiral operator (4.31).

[7]Strictly, the bound on α in SL(2) could be weaker. Here, we will use the bound in Liouville theory, which is obtained by studying the tube of the Coulomb branch.

Remarks on $Q_5 = 1$ The near horizon NS5/NS1 system does not seem to make sense for $Q_5 = 1$, since it uses an SU(2) current algebra at level $Q_5 - 2$ [6]. Hence in particular the above discussion does not make sense for $Q_5 = 1$. However, there is a sense in which one can approach $Q_5 = 1$: as reviewed in section 2, we can define a model that depends on an arbitrary positive integer N and move around in its moduli space until we reach the locus where the "small instanton singularity of $Q_5 = 1$" should appear. In this sense, something must happen as one approaches $Q_5 = 1$.

Though our derivation of (4.26) does not make sense for $Q_5 = 1$ (since the conformal field theory we used is not defined there), it is tempting to believe that the formula is still valid for that value of Q_5. If so, the gap vanishes at $Q_5 = 1$. We take this to mean that the ground state of the spacetime CFT becomes unnormalizable when one moves around in the parameter space (2.1) and approaches the locus of the $Q_5 = 1$ singularity. This is compatible intuitively with the argument at the end of section 2 indicating that the singularity is worse for $Q_5 = 1$, but we have no detailed interpretation to offer.

One might guess that the $Q_5 = 1$ system simply describes the scattering of low energy particles in spacetime from a single NS fivebrane in the weak coupling limit. If so, the essential difference between $Q_5 = 1$ and $Q_5 > 1$ is that (as there is [6] no tube for $Q_5 = 1$) the conformal field theory of a single fivebrane (unlike that of $Q_5 > 1$ coincident fivebranes) is nonsingular, and hence string perturbation theory is well behaved as the string coupling goes to zero. The weak coupling limit describes particles in spacetime that scatter from a potential (the fivebrane) but not from each other. Perhaps as one approaches the $Q_5 = 1$ singularity, the near horizon CFT somehow describes this physics. Hence, one may wonder whether the $Q_5 = 1$ system is related to the symmetric product $(\mathbb{R}^4 \times \mathbf{X})^{Q_1}/S_{Q_1}$. Clearly, this issue deserves better understanding.

5. Missing chiral states, and further applications

5.1 The fate of chiral primaries

The analysis of the worldsheet CFT of the NS1/NS5 theory in [19] has found chiral operators in the boundary conformal field theory with $\Delta \leq (Q_5 - 2)/2$. However, general considerations from a spacetime point of view [37] suggest that the system has chiral operators with $\Delta \leq Q_1 Q_5/2$. Where are the missing operators? The analysis of [37] is valid for generic values of the parameters, where the CFT is not singular. On the other hand, the analysis of [19] is valid on a sixteen-dimensional subspace of the system, where all the RR moduli of the NS1/NS5 system vanish. Precisely on this subspace, the system has, as we have seen, a continuum starting at $\Delta_0 = (Q_5 - 1)^2/4Q_5$. It is natural to suspect that the disappearance of some states when the RR moduli vanish are somehow associated with the appearance of this continuum or (shifting to a D1/D5 language) with the small instanton singularity.

Comparison to classical instanton moduli space To get some insight about this, let us look at the small instanton singularity classically. According to the ADHM construction, the moduli space of instantons on $\mathbb{R}^4$ is the Higgs branch of the following theory with eight supercharges: a SU(1) gauge theory with Q_5 hypermultiplets $\mathcal{A}^i$, $i = 1, \ldots, Q_5$ of charge 1. From the point of view of a theory with only four supercharges, each $\mathcal{A}^i$ splits as a pair of chiral multiplets A^i, B_i. The equations determining a vacuum are F-flatness

$$\sum_{i=1}^{Q_5} A^i B_i = 0 \tag{5.1}$$

and D-flatness

$$\sum_{i=1}^{Q_5} |A^i|^2 - \sum_{j=1}^{Q_5} |B_j|^2 = r\,. \tag{5.2}$$

Here we have included an FI interaction with coefficient r. For $r \neq 0$, the small instanton singularity is "blown up"; the singularity is recovered for $r = 0$. Up to an R-symmetry transformation (rotating the choice of four supercharges out of the eight), there is no loss of generality in assuming that the FI term appears only in (5.2) and that $r > 0$.

The small instanton moduli space $\mathcal{T}$ is the space of solutions of the above equations, modulo the SU(1) action $A \to e^{iw}A$, $B \to e^{-iw}B$. Setting $B = 0$, we see that $\mathcal{T}$ contains a copy of $U = \mathbf{CP}^{Q_5-1}$, of radius $\sqrt{r}$. $\mathcal{T}$ itself is the cotangent bundle $T^*U = T^*\mathbf{CP}^{Q_5-1}$ of U, as long as $r > 0$. For $r = 0$, the copy of $\mathbf{CP}^{Q_5-1}$ at the "center" of $\mathcal{T}$ is blown down to a point, and $\mathcal{T}$ has a conical singularity.

Now for topological reasons, for $r > 0$, there is an $\mathbf{L}^2$ harmonic form of the middle dimension on $\mathcal{T}$. Indeed, a delta function that is Poincaré dual to U is of compact support and also (since the fibration $T^*U \to U$ has nonzero Euler class) represents a nontrivial cohomology class. It can therefore be projected to a unique nonzero $\mathbf{L}^2$ form ω on $\mathcal{T}$. For $r > 0$, ω generates the image of the compactly supported cohomology of $\mathcal{T}$ in the ordinary cohomology of this space, and hence generates the $\mathbf{L}^2$ cohomology of $\mathcal{T}$.

Being Poincaré dual to a complex submanifold of complex codimension $Q_5 - 1$, ω is a form of degree $(Q_5 - 1, Q_5 - 1)$ on $\mathcal{T}$. It hence corresponds, for $r \neq 0$, to a chiral primary field of dimension $((Q_5 - 1)/2, (Q_5 - 1)/2)$ – exactly the quantum numbers of the first chiral primary that is "missing" in the conformal field theory analysis of the NS1/NS5 system without RR fields [19].

The $\mathbf{L}^2$ form ω has its support for A, B of order $\sqrt{r}$, since this is the radius of U. If, therefore, we approach the locus on which some chiral primaries are "missing" by taking $r \to 0$, then ω has delta function support at the singularity of the moduli space. (We use the phrase "delta function support" somewhat loosely to mean simply that as $r \to 0$, ω vanishes away from the conical singularity.) A more evocative way to say this is that as $r \to 0$, ω becomes concentrated at the singularity and disappears from the smooth part of $\mathcal{T}$.

Let us now try to interpret this in the conformal field theory. The fields A, B do not give a good description of the small instanton conformal field theory near the singularity. For this we should use instead the Liouville field ϕ of the long strings together with other degrees of freedom described in section 4. The small instanton region corresponds in that description to $\phi \to \infty$. For r very small and positive, an exponential coupling e^{ϕ} suppresses the region of large ϕ. The fact that the classical form ω has its support at $A, B \sim \sqrt{r}$ suggests that the chiral primary corresponding to ω in the (better) long string description has its support at, roughly, $e^{\phi} \sim 1/r^{\delta}$ for some $\delta > 0$. For $r \to 0$, this chiral primary state then disappears to $\phi = \infty$.

To prove that a chiral primary state corresponding to ω moves to $\phi = \infty$ as $r \to 0$, we would need a fuller understanding of the $\mathcal{N} = 4$ Liouville theory. But this behavior is certainly suggested by the classical facts that we have explained, and is entirely consistent with the fact that for $r = 0$, that is in the absence of RR fields, a chiral primary with the same quantum numbers as ω is missing in the NS1/NS5 conformal field theory.

The discussion so far makes it sound as if only one state should be "missing," but that is because we have focussed on just a single small instanton. Let us see what happens when we incorporate the other degrees of freedom. We let $\mathcal{M}_{Q_1,Q_5}$ be the moduli space of SU(Q_5) instantons of instanton number Q_1. The locus of $\mathcal{M}_{Q_1,Q_5}$ parametrizing configurations with a very small instanton looks like a product (more precisely, a fiber bundle) $\mathcal{T} \times \mathbf{X} \times \mathcal{M}_{Q_1-1,Q_5}$, where $\mathcal{M}_{Q_1-1,Q_5}$ parametrizes the $Q_1 - 1$ non-small instantons, $\mathbf{X}$ parametrizes the position of the small instanton, and $\mathcal{T}$ is the one-instanton moduli space discussed above. Let α be any harmonic form on $\mathbf{X} \times \mathcal{M}_{Q_1-1,Q_5}$. Then, when $\mathcal{T}$ is blown down by turning off the RR fields, $\omega \wedge \alpha$ has its support at the small instanton singularity in $\mathcal{T}$ (since ω does), so every cohomology class of this form disappears from the smooth part of the moduli space in this limit.

Harmonic forms on $\mathcal{M}_{Q_1,Q_5}$ of degree (p, q) with p (or q) less than $Q_5 - 1$ are supported, for $r \to 0$, on the smooth part of the moduli space (since the singular part does not support any $\mathbf{L}^2$ harmonic form of such degree). Hence, the chiral primaries with $j < (Q_5 - 1)/2$ should decay as one approaches the long string or small instanton region, as we found in section 4.

5.2 Further applications

We conclude by briefly remarking on some further applications.

In the NS1/NS5 case, the essence of our result concerns the behavior of long strings in the WZW model of SL$(2, \mathbf{R})$. This model has been studied over the years from many point of view, for instance as an example of a non-unitary CFT (see, e.g. [38, 39]), an example of string propagation in nontrivial and time-dependent backgrounds (see, e.g. [29, 40, 41] and references therein), as an ingredient in studying two-dimensional quantum gravity [42], and black holes [43], and in the context of the AdS/CFT correspondence [19]– [21]. This much-studied two-dimensional CFT has the strange behavior we have described: a continuum (above a certain threshold) coming from long strings, above and beyond the continuum coming from the noncompactness of the group manifold. This raises the interesting question, which we will not try to address, of what sort of singularities in correlation functions are generated by the long strings.

Breakdown of string perturbation theory In certain applications of conformal field theories of the sort we have been studying, the behavior we have found has dramatic consequences. The Type IIA string on **K3** is dual to the heterotic string on $\mathbf{T}^4$ [44, 45]; using this duality one can show that when Type IIA string theory is compactified on a space M with an A_N singularity, an enhanced A_N gauge symmetry occurs in the six dimensions orthogonal to M. This is a nonperturbative phenomenon that cannot be avoided by making the string coupling constant smaller. It thus represents a breakdown of string perturbation theory, possible only because of a singular behavior of the conformal field theory. This breakdown of perturbation theory is associated with an important role played by wrapped membranes. One might suspect that the breakdown of perturbation theory is associated with the appearance of a "tube" with a linearly growing coupling. However, the target space of the CFT does not exhibit such a singularity; indeed, the hyper-Kähler metric of the target space is subject to no quantum corrections. In fact, in some cases the target space has only an orbifold singularity and the theory differs from the non-singular orbifold only in the value of the θ angle [46].

Despite this, for string theory at the A_{n-1} singularity, a tubelike description arises in suitable variables via T-duality [8, 9]. These examples are similar to the ones we have been studying since, for example, the A_1 singularity is the small instanton singularity for $Q_5 = 2$. For the D1/D5 cases that we have studied in the present paper, we have seen a mechanism by which the CFT develops the expected tube in field space (but not in the original variables of the sigma model), and therefore the CFT is singular. Thus, our result can be seen as an analog or generalization of the tubelike description of the A_{n-1} singularities that comes from T-duality.

The D0/D4 system compactified on $\mathbf{X}$ is closely related to the D1/D5 system. Its low energy behavior is described by quantum mechanics whose target space is the moduli space of instantons on $\mathbf{X}$. For certain values of the parameters (one must set the three FI couplings to zero), this target space has singularities. They are associated either with small instantons or reduced structure group. The pathology discussed in this paper is a feature of $1+1$-dimensional conformal field theory and does not affect D0/D4 quantum mechanics, which remains well behaved even with the target space develops singularities. However, some low energy states in the quantum mechanics are supported near the singularities, for reasons explained in section 5.1. In making this assertion, we are assuming that, when one turns off the FI couplings, the states in question do not spread on the Coulomb branch. Assuming this, these states are a generalization, to higher Q_5, of the D0/D4 bound state at threshold whose existence was proved in [47].

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JHEP04(1999)017

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